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SECTIONAL CURVATURES OF THE SIEGEL-JACOBI SPACE

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Abstract. In this paper, we compute the sectional curvatures and the scalar curvature of the Siegel-Jacobi space $\mathbb{H}_1 \times \mathbb{C}$ of degree 1 and index 1 explicitly.

1. Introduction

For a given fixed positive integer n, we let

$$\mathbb{H}_n := \{\, Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ,\ \operatorname{Im} Z > 0\}$$

be the Siegel upper half plane of degree n and let

$$Sp(n, \mathbb{R}) = \{M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n\}$$

be the symplectic group of degree n, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transpose of a matrix M, $\operatorname{Im} Z$ denotes the imaginary part of Z and

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Here I_n denotes the $n \times n$ identity matrix. It is easy to see that $Sp(n, \mathbb{R})$ acts on $\mathbb{H}_n$ transitively by

$$M \cdot Z := (AZ + B)(CZ + D)^{-1}, \tag{1.1}$$

where $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n, \mathbb{R})$ and $Z \in \mathbb{H}_n$.

For two positive integers n and m, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} := \{\, (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric}\}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') := (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

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We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G_{n,m}^J := Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\Big(M,(\lambda,\mu;\kappa)\Big)\cdot\Big(M',(\lambda',\mu';\kappa')\Big) := \Big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda')\Big)$$

with $M, M' \in Sp(n,\mathbb{R}), (\lambda,\mu;\kappa),\ (\lambda',\mu';\kappa') \in H_{\mathbb{R}}^{(n,m)}$ and $(\tilde{\lambda},\tilde{\mu}) := (\lambda,\mu)M'$.

We call this group $G_{n,m}^J$ the *Jacobi group* of degree n and index m. It is easy to see that $G_{n,m}^J$ acts on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$\Big(M,(\lambda,\mu;\kappa)\Big)\cdot(Z,W) := \Big(M\cdot Z,\ (W+\lambda Z+\mu)(CZ+D)^{-1}\Big), \tag{1.2}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$.

The homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ is called the *Siegel-Jacobi space* of degree n and index m. We refer to [3, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] for more details on materials related to the Siegel-Jacobi space.

In [14], the author proved that for any two positive real numbers A and B, the following metric

$$\begin{aligned} ds^2_{n,m;A,B} &= A\,\sigma\Big(Y^{-1}dZ\,Y^{-1}d\overline{Z}\Big) \\ &\quad + B\,\Big\{\sigma\Big(Y^{-1}\,{}^tV\,V\,Y^{-1}dZ\,Y^{-1}d\overline{Z}\Big) + \sigma\Big(Y^{-1}\,{}^t(dW)\,d\overline{W}\Big) \\ &\quad - \sigma\Big(V\,Y^{-1}dZ\,Y^{-1}\,{}^t(d\overline{W})\Big) - \sigma\Big(V\,Y^{-1}d\overline{Z}\,Y^{-1}\,{}^t(dW)\Big)\Big\} \end{aligned} \tag{1.3}$$

is a Riemannian metric on the Siegel-Jacobi space $\mathbb{H}_n\times\mathbb{C}^{(m,n)}$ which is invariant under the action (1.2) of the Jacobi group $G_{n,m}^J$, where $Z = X+i\,Y \in \mathbb{H}_n$, $W = U+i\,V \in \mathbb{C}^{(m,n)}$ with $Z=(z_{ij}),\ \ W=(w_{kl})$ and X,Y,U,V real, we put

$$dZ=(dz_{ij}),\ \ d\overline{Z}=(d\overline{z}_{ij}),\ \ dW=(dw_{kl}),\ \ d\overline{W}=(d\overline{w}_{kl})$$

and $\sigma(A)$ denotes the trace of a square matrix A. Also he computed the Laplace-Beltrami operator of the Siegel-Jacobi space $(\mathbb{H}_n \times \mathbb{C}^{(m,n)}, ds^2_{n,m;A,B})$ explicitly.

In this paper, we consider the case $n=1$ and $m=1$. In this case, we have a Riemannian metric

$$\begin{aligned} &ds^2_{1,1;A,B} \\ &= A\,\frac{dx^2+dy^2}{y^2} + B\,\Big\{\,\frac{v^2}{y^3}\left(dx^2+dy^2\right) + \frac{1}{y}\left(du^2+dv^2\right) - \frac{2v}{y^2}\left(dxdu+dydv\right)\Big\} \end{aligned} \tag{1.4}$$

on $\mathbb{H}_1\times\mathbb{C}$ which is invariant under the action (1.2) of the Jacobi group $G_{1,1}^J = SL(2,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(1,1)}$, where $z=x+i\,y \in \mathbb{H}_1$ and $w = u+i\,v\in\mathbb{C}$ with x,y,u,v real coordinates. We also refer to [1] and [4] for the metric (1.4). According to

Theorem 1.2 in [14], we see that the Laplace-Beltrami operator $\Delta_{1,1;A,B}$ of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\ ds^2_{1,1;A,B})$ is given by

$$\Delta_{1,1;A,B} = \frac{1}{A}\left\{ y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right) + 2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right)\right\} + \frac{y}{B}\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right). \tag{1.5}$$

The purpose of this paper is to compute the sectional curvatures of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\ ds^2_{1,1;A,B})$ explicitly. We will prove that the scalar curvature $r(p)$ of $(\mathbb{H}_1 \times \mathbb{C},\ ds^2_{1,1;A,B})$ is constant, precisely, $r(p) = -\frac{3}{A}$ for all $p \in \mathbb{H}_1 \times \mathbb{C}$.

This paper is organized as follows. In Section 2, we compute the Christoffel symbols Γ^k_{ij} of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C}, ds^2_{1,1;A,B})$ explicitly. In Section 3, we compute the sectional curvatures of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C}, ds^2_{1,1;A,B})$ explicitly. We prove that the scalar curvature of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C}, ds^2_{1,1;A,B})$ is given by $-\frac{3}{A}$ and that the scalar curvature is independent of the choice of B. In the final section, we discuss the invariant Riemannian metrics of the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ and their Laplace-Beltrami operators.

Notations: We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose matrix of M. I_n denotes the identity matrix of degree n.

2. Preliminaries

For brevity, we write $M := \mathbb{H}_1 \times \mathbb{C}$. Then M is a four dimensional Riemannian manifold with a metric ds^2 given by (1.4). We denote by $C^\infty(M)$ and $\mathcal{X}(M)$ be the algebra of all C^∞ functions on M and the algebra of all C^∞ vector fields on M respectively. It is well known that there exists a uniquely determined Riemannian connection ∇ on M (cf. [2], p. 314). That is, the connection ∇ is a mapping $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$, denoted by $\nabla(X,Y) = \nabla_X Y$ which satisfies the following properties (R1)-(R4): For all $f, g \in C^\infty(M)$ and $X, Y, Z, W \in \mathcal{X}(M)$,

$$(R1)\qquad \nabla_{fX+gY}Z = f(\nabla_X Z) + g(\nabla_Y Z),$$

$$(R2)\qquad \nabla_X(fY+gZ) = f(\nabla_X Y) + g(\nabla_X Z) + (Xf)Y + (Xg)Z,$$

$$(R3)\qquad [X,Y] = \nabla_X Y - \nabla_Y X \ \text{(symmetry)}, \ \text{and}$$

$$(R4)\qquad X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

where $g(Y,Z)$ denoted the inner product determined by the Riemannian metric ds^2 on M.

Now we fix a local cooodinate x, y, u, v with $z = x + i\,y$ and $w = u + i\,v$. Then the smooth vector fields

$$E_1 := \frac{\partial}{\partial x}, \quad E_2 := \frac{\partial}{\partial y}, \quad E_3 := \frac{\partial}{\partial u} \quad \text{and} \quad E_4 := \frac{\partial}{\partial v}$$

form a local frame fields on M. We recall that the *Christoffel symbols* Γ^k_{ij} $(1 \le i, j, k \le 4)$ are defined by

$$\nabla_{E_i} E_j := \sum_{k=1}^{4} \Gamma^k_{ij}\, E_k, \quad 1 \le i, j \le 4. \tag{2.1}$$

According to (1.4), the matrix form $g = (g_{ij})$ of the metric $ds^2_{1,1;A,B}$ is of the form

$$g = (g_{ij}) = \begin{pmatrix} \dfrac{Ay+Bv^2}{y^3} & 0 & -\dfrac{Bv}{y^2} & 0 \\ 0 & \dfrac{Ay+Bv^2}{y^3} & 0 & -\dfrac{Bv}{y^2} \\ -\dfrac{Bv}{y^2} & 0 & \dfrac{B}{y} & 0 \\ 0 & -\dfrac{Bv}{y^2} & 0 & \dfrac{B}{y} \end{pmatrix}. \tag{2.2}$$

Then it is easy to see that $\det(g_{ij}) = A^2 B^2\, y^{-6}$ and the inverse matrix $g^{-1} := (g^{ij})$ of $g = (g_{ij})$ is given by

$$g^{-1} = \left(g^{ij}\right) = \begin{pmatrix} \dfrac{y^2}{A} & 0 & \dfrac{yv}{A} & 0 \\ 0 & \dfrac{y^2}{A} & 0 & \dfrac{yv}{A} \\ \dfrac{yv}{A} & 0 & \dfrac{Ay+Bv^2}{AB} & 0 \\ 0 & \dfrac{yv}{A} & 0 & \dfrac{Ay+Bv^2}{AB} \end{pmatrix}. \tag{2.3}$$

Lemma 2.1. *For all $i, j, k, \Gamma^k_{ij} = \Gamma^k_{ji}$. The Christoffel symbols Γ^k_{ij}'s $(1 \le i, j, k \le 4)$ are given by*

$$\Gamma^2_{11} = \frac{2A\,y + Bv^2}{2A\,y^2}, \qquad \Gamma^1_{12} = \Gamma^2_{22} = -\frac{2Ay + Bv^2}{2Ay^2}$$

$$\Gamma^4_{11} = \frac{Bv^3}{2Ay^3}, \qquad \Gamma^3_{12} = \Gamma^4_{22} = -\frac{Bv^3}{2Ay^3}$$

$$\Gamma^1_{14} = \Gamma^1_{23} = \Gamma^2_{24} = \Gamma^4_{33} = \frac{Bv}{2Ay}, \qquad \Gamma^2_{13} = \Gamma^3_{34} = \Gamma^4_{44} = -\frac{Bv}{2Ay}$$

$$\Gamma^4_{13} = \frac{Ay - Bv^2}{2Ay^2}, \qquad \Gamma^3_{14} = \Gamma^3_{23} = \Gamma^4_{24} = -\frac{Ay - Bv^2}{2Ay^2}$$

$$\Gamma^2_{33} = \frac{B}{2A}, \qquad \Gamma^2_{44} = \Gamma^1_{34} = -\frac{B}{2A}$$

and all other $\Gamma^k_{ij} = 0$.

Proof. The first statement follows immediately from the symmetry relation (R3). We recall (cf. [2], p. 318 or [8], p. 210) that

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{s=1}^{4} g^{ks}(E_j g_{si} - E_s g_{ij} + E_i g_{js}) \tag{2.4}$$

for all i, j, k. By an easy computation, we get all Γ^k_{ij}. □

We define the functions

$$h_A := \frac{y^{\frac{3}{2}}}{(Ay + Bv^2)^{\frac{1}{2}}}, \qquad h_B := \frac{\sqrt{B}\, yv}{\sqrt{A}\,(Ay + Bv^2)^{\frac{1}{2}}}, \qquad h_C := \frac{(y + v^2)^{\frac{1}{2}}}{\sqrt{AB}}. \tag{2.5}$$

An easy computation gives the following:

Lemma 2.2.

$$\frac{\partial h_A}{\partial y} = \frac{y^{\frac{1}{2}}(2Ay + 3Bv^2)}{2(Ay + Bv^2)^{\frac{3}{2}}}, \qquad \frac{\partial h_B}{\partial y} = \frac{\sqrt{B}\, v(Ay + 2Bv^2)}{2\sqrt{A}\,(Ay + Bv^2)^{\frac{3}{2}}},$$

$$\frac{\partial h_C}{\partial y} = \frac{\sqrt{A}}{2\sqrt{B}\,(Ay + Bv^2)^{\frac{1}{2}}}, \qquad \frac{\partial h_A}{\partial v} = -\frac{B\, y^{\frac{3}{2}} v}{(Ay + Bv^2)^{\frac{3}{2}}},$$

$$\frac{\partial h_B}{\partial v} = \frac{\sqrt{AB}\, y^2}{(Ay + Bv^2)^{\frac{3}{2}}}, \qquad \frac{\partial h_C}{\partial v} = \frac{\sqrt{B}\, v}{\sqrt{A}\,(Ay + Bv^2)^{\frac{1}{2}}}$$

and

$$\frac{\partial h_A}{\partial x} = \frac{\partial h_B}{\partial x} = \frac{\partial h_C}{\partial x} = \frac{\partial h_A}{\partial u} = \frac{\partial h_B}{\partial u} = \frac{\partial h_C}{\partial u} = 0.$$

Lemma 2.3. *The following frame field* F_1, F_2, F_3, F_4 *defined by*

$$F_1 := h_A E_1, \qquad F_2 := h_A E_2$$
$$F_3 := h_B E_1 + h_C E_3, \qquad F_4 := h_B E_2 + h_C E_4$$

form an orthonomal frame field on M. And they satisfy the following relations

$$[F_1, F_2] = -\frac{y^2(2Ay + 3Bv^2)}{2(Ay + Bv^2)^2}\, E_1, \qquad [F_1, F_3] = 0,$$

$$[F_1, F_4] = -\frac{B\sqrt{B}\, y^{\frac{3}{2}} v^3}{2\sqrt{A}\,(Ay + Bv^2)^2}\, E_1,$$

$$[F_2, F_3] = \frac{\sqrt{B}\, y^{\frac{3}{2}} v(Ay + 2Bv^2)}{2\sqrt{A}\,(Ay + Bv^2)^2}\, E_1 + \frac{\sqrt{A}\, y^{\frac{3}{2}}}{2\sqrt{B}\,(Ay + Bv^2)}\, E_3,$$

$$[F_2, F_4] = \frac{\sqrt{B}\, y^{\frac{3}{2}} v}{2\sqrt{A}\,(Ay + Bv^2)}\, E_2 \;+\; \frac{\sqrt{A}\, y^{\frac{3}{2}}}{2\sqrt{B}\,(Ay + Bv^2)}\, E_4$$

and

$$[F_3, F_4] = -\frac{2A^2y^3 + 3ABy^2v^2 + 2B^2yv^4}{2A(Ay + Bv^2)^2}\, E_1 \;-\; \frac{3Ayv + 2Bv^3}{2A(Ay + Bv^2)}\, E_3.$$

Proof. The first statement follows from the Gram-Schmidt orthogonalization process. The proof of the second statement follows from a direct computation. □

Definition 2.1. Let X and Y be two smooth vector fields on M. The curvature operator $R(X, Y) : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ is defined as

$$R(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z, \quad Z \in \mathcal{X}(M). \tag{2.6}$$

For a quadruple (X,Y,Z,W) of smooth vector fields on M, we define

$$R(X, Y, Z, W) := g(R(X, Y)Z, W). \tag{2.7}$$

The tensor $R(X, Y, Z, W)$ is called the *Riemann curvature tensor* of M.

3. Sectional curvatures

For any point $p \in M$, we let $\pi_{X,Y}$ be the plane section of tangent space $T_p(M)$ of M at p spanned by two orthonormal tangent vectors X and Y in $T_p(M)$. We recall that the *sectional curvature* $K_p(\pi_{X,Y})$ of $\pi_{X,Y}$ is defined by

$$K_p(\pi_{X,Y}) := -R(X, Y, Z, W) = -g(R(X, Y)Z, W), \tag{3.1}$$

where $R(X, Y, Z, W)$ denotes the Riemann curvature tensor of M. In fact, the sectional curvature $K_p(\pi_{X,Y})$ is independent of the choice of two orthonormal basis of the section $\pi_{X,Y}$.

Theorem 3.1. *For any point $p = (x, y, u, v) \in M$, we let π_{ij} the plane section of $T_p(M)$ spanned by two orthonormal vectors F_{ip} and F_{jp} of $T_p(M)$. Then the sectional curvatures $K_p(\pi_{X,Y})$ are given by*

$$K_p(\pi_{12}) = -\frac{1}{A} + \frac{3B^2v^4}{2A\,(Ay + Bv^2)^2}, \qquad K_p(\pi_{13}) = -\frac{1}{4A},$$

$$K_p(\pi_{14}) = -\frac{1}{4A} + \frac{3AByv^2}{2A\,(Ay + Bv^2)^2}, \qquad K_p(\pi_{23}) = -\frac{1}{4A} + \frac{3AByv^2}{2A\,(Ay + Bv^2)^2},$$

$$K_p(\pi_{24}) = -\frac{1}{4A}, \qquad K_p(\pi_{34}) = \frac{1}{2A} - \frac{3Bv^2(2Ay + Bv^2)}{2A\,(Ay + Bv^2)^2}.$$

Proof. We observe that $K_p(\pi_{ij}) = -g(R(F_{ip}, F_{jp})F_{ip}, F_{jp})$ for $1 \le i, j \le 4$. By a direct computation, we obtain

$$\nabla_{E_1}\nabla_{E_2}E_1 = (\,\Gamma^2_{11}\Gamma^1_{12} + \Gamma^3_{12}\Gamma^2_{13}\,)\,E_2 \;+\; (\,\Gamma^4_{11}\Gamma^1_{12} + \Gamma^3_{12}\Gamma^4_{13}\,)\,E_4,$$

$$\nabla_{E_1}\nabla_{E_2}E_2 = (\,\Gamma^1_{12}\Gamma^2_{22} + \Gamma^1_{14}\Gamma^4_{22}\,)\,E_1 \;+\; (\,\Gamma^3_{12}\Gamma^2_{22} + \Gamma^3_{14}\Gamma^4_{22}\,)\,E_3,$$

$$\nabla_{E_1}\nabla_{E_2}E_3 = (\,\Gamma^2_{11}\Gamma^1_{23} \,+\, \Gamma^2_{13}\Gamma^2_{23}\,)\,E_2 \;+\; (\,\Gamma^4_{11}\Gamma^1_{23} \,+\, \Gamma^4_{13}\Gamma^3_{23}\,)\,E_4,$$
$$\nabla_{E_1}\nabla_{E_3}E_1 = (\,\Gamma^1_{12}\Gamma^2_{13} \,+\, \Gamma^4_{13}\Gamma^1_{14}\,)\,E_1 \;+\; (\,\Gamma^3_{12}\Gamma^2_{13} \,+\, \Gamma^4_{13}\Gamma^3_{14}\,)\,E_3,$$
$$\nabla_{E_1}\nabla_{E_4}E_1 = (\,\Gamma^2_{11}\Gamma^1_{14} \,+\, \Gamma^2_{13}\Gamma^3_{14}\,)\,E_2 \;+\; (\,\Gamma^4_{11}\Gamma^1_{14} \,+\, \Gamma^4_{13}\Gamma^3_{14}\,)\,E_4,$$
$$\nabla_{E_1}\nabla_{E_4}E_3 = (\,\Gamma^2_{11}\Gamma^1_{12} \,+\, \Gamma^3_{12}\Gamma^2_{13}\,)\,E_2 \;+\; (\,\Gamma^4_{11}\Gamma^1_{12} \,+\, \Gamma^3_{12}\Gamma^4_{13}\,)\,E_4,$$
$$\nabla_{E_2}\nabla_{E_1}E_1 = \left(\Gamma^2_{11}\Gamma^2_{22} + \Gamma^4_{11}\Gamma^2_{24} + \tfrac{\partial\Gamma^2_{11}}{\partial y}\right)E_2 + \left(\Gamma^2_{11}\Gamma^4_{22} + \Gamma^4_{11}\Gamma^4_{24} + \tfrac{\partial\Gamma^4_{11}}{\partial y}\right)E_4,$$
$$\nabla_{E_2}\nabla_{E_1}E_2 = \left(\Gamma^1_{12}\Gamma^1_{12} + \Gamma^3_{12}\Gamma^1_{23} + \tfrac{\partial\Gamma^1_{12}}{\partial y}\right)E_1 + \left(\Gamma^1_{12}\Gamma^3_{12} + \Gamma^3_{12}\Gamma^3_{23} + \tfrac{\partial\Gamma^3_{12}}{\partial y}\right)E_3,$$
$$\nabla_{E_2}\nabla_{E_1}E_3 = \left(\Gamma^2_{13}\Gamma^2_{22} + \Gamma^4_{13}\Gamma^2_{24} + \tfrac{\partial\Gamma^2_{13}}{\partial y}\right)E_2 + \left(\Gamma^2_{13}\Gamma^4_{22} + \Gamma^4_{13}\Gamma^4_{24} + \tfrac{\partial\Gamma^4_{13}}{\partial y}\right)E_4,$$
$$\nabla_{E_2}\nabla_{E_3}E_2 = \left(\Gamma^1_{12}\Gamma^1_{23} + \Gamma^1_{32}\Gamma^3_{23} + \tfrac{\partial\Gamma^1_{23}}{\partial y}\right)E_1 + \left(\Gamma^3_{12}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^3_{23} + \tfrac{\partial\Gamma^3_{23}}{\partial y}\right)E_3,$$
$$\nabla_{E_2}\nabla_{E_3}E_3 = \left(\Gamma^2_{22}\Gamma^2_{33} + \Gamma^2_{24}\Gamma^4_{33} + \tfrac{\partial\Gamma^2_{33}}{\partial y}\right)E_2 + \left(\Gamma^4_{22}\Gamma^2_{33} + \Gamma^4_{24}\Gamma^4_{33} + \tfrac{\partial\Gamma^4_{33}}{\partial y}\right)E_4,$$
$$\nabla_{E_2}\nabla_{E_4}E_2 = \left(\Gamma^2_{22}\Gamma^2_{24} + \Gamma^2_{24}\Gamma^4_{24} + \tfrac{\partial\Gamma^2_{24}}{\partial y}\right)E_2 + \left(\Gamma^4_{22}\Gamma^2_{24} + \Gamma^4_{24}\Gamma^4_{24} + \tfrac{\partial\Gamma^4_{24}}{\partial y}\right)E_4,$$
$$\nabla_{E_3}\nabla_{E_1}E_1 = (\,\Gamma^2_{11}\Gamma^1_{23} + \Gamma^4_{11}\Gamma^1_{34}\,)\,E_1 + (\,\Gamma^2_{11}\Gamma^3_{23} + \Gamma^4_{11}\Gamma^3_{34}\,)\,E_3,$$
$$\nabla_{E_3}\nabla_{E_2}E_1 = (\,\Gamma^1_{12}\Gamma^2_{13} + \Gamma^3_{12}\Gamma^2_{33}\,)\,E_2 + (\,\Gamma^1_{12}\Gamma^4_{13} + \Gamma^3_{12}\Gamma^4_{33}\,)\,E_4,$$
$$\nabla_{E_3}\nabla_{E_2}E_2 = (\,\Gamma^2_{22}\Gamma^1_{23} + \Gamma^4_{22}\Gamma^1_{34}\,)\,E_1 + (\,\Gamma^2_{22}\Gamma^3_{23} + \Gamma^4_{22}\Gamma^3_{34}\,)\,E_3,$$
$$\nabla_{E_3}\nabla_{E_2}E_3 = (\,\Gamma^2_{13}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^2_{33}\,)\,E_2 + (\,\Gamma^4_{13}\Gamma^1_{23} + \Gamma^3_{23}\Gamma^4_{33}\,)\,E_4,$$
$$\nabla_{E_3}\nabla_{E_4}E_1 = (\,\Gamma^2_{13}\Gamma^1_{14} + \Gamma^3_{14}\Gamma^2_{33}\,)\,E_2 + (\,\Gamma^4_{13}\Gamma^1_{14} + \Gamma^3_{14}\Gamma^4_{33}\,)\,E_4,$$
$$\nabla_{E_3}\nabla_{E_4}E_3 = (\,\Gamma^2_{13}\Gamma^1_{34} + \Gamma^2_{33}\Gamma^3_{34}\,)\,E_2 + (\,\Gamma^4_{13}\Gamma^1_{34} \,+\, \Gamma^4_{33}\Gamma^3_{34}\,)\,E_4,$$
$$\nabla_{E_4}\nabla_{E_1}E_1 = \left(\Gamma^2_{11}\Gamma^2_{24} + \Gamma^4_{11}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{11}}{\partial v}\right)E_2 + \left(\Gamma^2_{11}\Gamma^4_{24} + \Gamma^4_{11}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{11}}{\partial v}\right)E_4,$$
$$\nabla_{E_4}\nabla_{E_1}E_3 = \left(\Gamma^2_{13}\Gamma^2_{24} + \Gamma^4_{13}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{13}}{\partial v}\right)E_2 + \left(\Gamma^2_{13}\Gamma^4_{24} + \Gamma^4_{13}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{13}}{\partial v}\right)E_4,$$
$$\nabla_{E_4}\nabla_{E_2}E_2 = \left(\Gamma^2_{22}\Gamma^2_{24} + \Gamma^4_{22}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{22}}{\partial v}\right)E_2 + \left(\Gamma^2_{22}\Gamma^4_{24} + \Gamma^4_{22}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{22}}{\partial v}\right)E_4,$$
$$\nabla_{E_4}\nabla_{E_3}E_3 = \left(\Gamma^2_{24}\Gamma^2_{33} + \Gamma^4_{33}\Gamma^2_{44} + \tfrac{\partial\Gamma^2_{33}}{\partial v}\right)E_2 + \left(\Gamma^2_{33}\Gamma^4_{24} + \Gamma^4_{33}\Gamma^4_{44} + \tfrac{\partial\Gamma^4_{33}}{\partial v}\right)E_4.$$

Thus according to Lemma 2.2, Lemma 2.3 and the above formulas, we have

$$\begin{aligned} R(F_1,F_2)F_1 = &-h_A\left\{\left(h_A\tfrac{\partial h_A}{\partial y}+\theta_1\right)\Gamma^2_{11} + h_A^2\tfrac{\partial\Gamma^2_{11}}{\partial y}\right\}E_2 \\ &-h_A\left\{\left(h_A\tfrac{\partial h_A}{\partial y}+\theta_1\right)\Gamma^4_{11} + h_A^2\tfrac{\partial\Gamma^4_{11}}{\partial y}\right\}E_4, \end{aligned}$$

$$\begin{aligned} R(F_1,F_3)F_1 = &\, h_A^2h_C\left\{\left(\Gamma^1_{14}\Gamma^4_{13} - \Gamma^4_{11}\Gamma^1_{34}\right)\right\}E_1 \\ &+ h_A^2h_C\left\{\left(\Gamma^3_{12}\Gamma^2_{13} + \Gamma^4_{13}\Gamma^3_{14} - \Gamma^2_{11}\Gamma^3_{32} - \Gamma^4_{11}\Gamma^3_{34}\right)\right\}E_3, \end{aligned}$$

$$
\begin{aligned}
R(F_1,F_4)F_1 = h_A\Big\{ & h_A h_C\left(\Gamma_{13}^2\Gamma_{14}^3 - \Gamma_{11}^4\Gamma_{44}^2 - \tfrac{\partial\Gamma_{11}^2}{\partial v}\right)\\
& - h_A h_B\tfrac{\partial\Gamma_{11}^2}{\partial y} - \left(h_B\tfrac{\partial h_A}{\partial y} + h_C\tfrac{\partial h_A}{\partial v} + \theta_2\right)\Gamma_{11}^2\Big\}E_2\\
& + h_A\Big\{h_A h_C\left(\Gamma_{13}^4\Gamma_{14}^3 + \Gamma_{11}^4\Gamma_{14}^1 - \Gamma_{11}^2\Gamma_{24}^2 - \Gamma_{11}^4\Gamma_{44}^4 - \tfrac{\partial\Gamma_{11}^4}{\partial v}\right)\\
& - h_A h_B\tfrac{\partial\Gamma_{11}^4}{\partial y} - \left(h_B\tfrac{\partial h_A}{\partial y} + h_C\tfrac{\partial h_A}{\partial v} + \theta_2\right)\Gamma_{11}^4\Big\}E_4,
\end{aligned}
$$

$$
\begin{aligned}
R(F_2,F_3)F_2 = h_A\Big\{ & h_A h_C\left(\Gamma_{23}^1\Gamma_{23}^3 - \Gamma_{22}^4\Gamma_{34}^1 + \tfrac{\partial\Gamma_{23}^1}{\partial y}\right)\\
& + \left(h_A h_B\tfrac{\partial\Gamma_{12}^1}{\partial y} + h_A\tfrac{\partial h_B}{\partial y}\Gamma_{12}^1 + h_A\tfrac{\partial h_C}{\partial y}\Gamma_{23}^1 - \theta_3\Gamma_{12}^1 - \theta_4\Gamma_{23}^1\right)\Big\}E_1\\
& + h_A\Big\{h_A h_C\left(\Gamma_{12}^3\Gamma_{23}^1 + \Gamma_{23}^3\Gamma_{23}^3 - \Gamma_{22}^2\Gamma_{23}^3 - \Gamma_{22}^4\Gamma_{34}^3 + \tfrac{\partial\Gamma_{23}^3}{\partial y}\right)\\
& + \left(h_A h_B\tfrac{\partial\Gamma_{12}^3}{\partial y} + h_A\tfrac{\partial h_B}{\partial y}\Gamma_{12}^3 + h_A\tfrac{\partial h_C}{\partial y}\Gamma_{23}^3 - \theta_3\Gamma_{12}^3 - \theta_4\Gamma_{23}^3\right)\Big\}E_3,
\end{aligned}
$$

$$
\begin{aligned}
R(F_2,F_4)F_2 = \Big\{ & h_A^2\, h_C\left(\Gamma_{24}^2\Gamma_{24}^4 - \Gamma_{22}^4\Gamma_{44}^2 + \tfrac{\partial\Gamma_{24}^2}{\partial y} - \tfrac{\partial\Gamma_{22}^2}{\partial v}\right)\\
& + h_A\left(h_A\tfrac{\partial h_B}{\partial y} - h_B\tfrac{\partial h_A}{\partial y} - h_C\tfrac{\partial h_A}{\partial v}\right)\Gamma_{22}^2 + h_A^2\tfrac{\partial h_C}{\partial y}\Gamma_{24}^2\\
& + h_A\tfrac{\partial h_A}{\partial y}\tfrac{\partial h_B}{\partial y} + h_A\tfrac{\partial h_C}{\partial y}\tfrac{\partial h_A}{\partial v} - h_B\left(\tfrac{\partial h_A}{\partial y}\right)^2 - h_C\tfrac{\partial h_A}{\partial y}\tfrac{\partial h_A}{\partial v}\\
& - h_A\theta_5\Gamma_{22}^2 - h_A\theta_4\Gamma_{24}^2 - \theta_5\tfrac{\partial h_A}{\partial y} - \theta_4\tfrac{\partial h_A}{\partial v}\Big\}E_2\\
& + \Big\{h_A^2\, h_C\left(\Gamma_{22}^4\Gamma_{24}^2 + \Gamma_{24}^4\Gamma_{24}^4 + \tfrac{\partial\Gamma_{24}^4}{\partial y} - \Gamma_{22}^2\Gamma_{24}^4 - \Gamma_{22}^4\Gamma_{44}^4\right.\\
& \left. - \tfrac{\partial\Gamma_{22}^4}{\partial v}\right) + h_A\left(h_A\tfrac{\partial h_B}{\partial y} - h_B\tfrac{\partial h_A}{\partial y} - h_C\tfrac{\partial h_A}{\partial v}\right)\Gamma_{22}^4\\
& + h_A^2\tfrac{\partial h_C}{\partial y}\Gamma_{24}^4 - h_A\theta_5\Gamma_{22}^4 - h_A\theta_4\Gamma_{24}^4\Big\}E_4,
\end{aligned}
$$

$$
R(F_3,F_4)F_3 = -\Big\{h_B^2\left(h_B\tfrac{\partial\Gamma_{11}^2}{\partial y} + h_C\tfrac{\partial\Gamma_{11}^2}{\partial v} + 2h_C\tfrac{\partial\Gamma_{13}^2}{\partial y}\right)
$$

$$+\, h_C^2 \left(h_C \frac{\partial \Gamma_{33}^2}{\partial v} + h_B \frac{\partial \Gamma_{33}^2}{\partial v} + 2h_B \frac{\partial \Gamma_{13}^2}{\partial v} \right) \Bigg\} E_2$$

$$-\Bigg\{ h_B^2 \left(h_B \frac{\partial \Gamma_{11}^4}{\partial y} + h_C \frac{\partial \Gamma_{11}^4}{\partial v} + 2h_C \frac{\partial \Gamma_{13}^4}{\partial y} \right)$$

$$+\, h_C^2 \left(h_C \frac{\partial \Gamma_{33}^4}{\partial v} + h_B \frac{\partial \Gamma_{33}^4}{\partial v} + 2h_B \frac{\partial \Gamma_{13}^4}{\partial v} \right) \Bigg\} E_4,$$

where we put

$$\theta_1 := -\frac{y^2\,(2Ay+3Bv^2)}{2\,(Ay+Bv^2)^2},\ \theta_2 := -\frac{B\sqrt{B}\,y^{\frac{3}{2}}v^3}{2\sqrt{A}\,(Ay+Bv^2)^2},\ \theta_3 := \frac{\sqrt{B}\,y^{\frac{3}{2}}v(Ay+2Bv^2)}{2\sqrt{A}\,(Ay+Bv^2)^2}$$

and

$$\theta_4 := \frac{\sqrt{A}\,y^{\frac{3}{2}}}{2\sqrt{B}\,(Ay+Bv^2)}, \qquad \theta_5 := \frac{\sqrt{B}\,y^{\frac{3}{2}}v}{2\sqrt{A}\,(Ay+Bv^2)}.$$

Using (2.2), (2.5), (2.7), Lemma 2.1, Lemma 2.2 and the above formulas, we obtain the above sectional curvatures $K_p(\pi_{ij})$ for $1 \le i \le j \le 4$. □

Theorem 3.2. *The scalar curvature $r(p)$ of the Siegel-Jacobi space*

$$(M, ds^2_{1,1;A,B})$$

is

$$r(p) = -\frac{3}{A} \quad \textit{for all } p \in M.$$

Proof. We recall that the scalar curvature $r(p)$ of M is defined as

$$r(p) := \sum_{i,j=1}^{4} R\,(F_{ip}, F_{jp}, F_{jp}, F_{ip})\,, \quad p \in M.$$

We note that the scalar curvature $r(p)$ is independent of the choice of an orthonormal basis of $T_p(M)$. Since the following symmetry relations

$$R(X,Y)Z + R(Y,X)Z = 0$$

hold for all $X, Y, Z \in \mathcal{X}(M)$, we have

$$\begin{aligned} r(p) = &-2\Big\{ R\,(F_{1p}, F_{2p}, F_{1p}, F_{2p}) + R\,(F_{1p}, F_{3p}, F_{1p}, F_{3p}) \\ &+ R\,(F_{1p}, F_{4p}, F_{1p}, F_{4p}) + R\,(F_{2p}, F_{3p}, F_{2p}, F_{3p}) \\ &+ R\,(F_{2p}, F_{4p}, F_{2p}, F_{4p}) + R\,(F_{3p}, F_{4p}, F_{3p}, F_{4p}) \Big\}. \end{aligned}$$

Accoding to Theorem 3.1, we obtain

$$r(p) = -\frac{3}{A}.$$

This completes the proof of the above theorem. □

Remark 3.1. The Poincaré upper half plane $\mathbb{H}_1$ is a two dimensional Riemannian manifold with the Poincaré metric

$$ds_0^2 := \frac{dx^2 + dy^2}{y^2}, \quad z = x + i\,y \in \mathbb{H}_1 \text{ with } x, y \text{ real.}$$

It is easily seen that the Gaussian curvature of $(\mathbb{H}_1,\, ds_0^2)$ is -1 everywhere and $(\mathbb{H}_1,\, ds_0^2)$ is an Einstein manifold. Indeed, if we denote by $S_0(X,Y)$ the Ricci curvature of $(\mathbb{H}_1,\, ds_0^2)$, then we have

$$S_0(X,Y) = -g_0(X,Y) \qquad \text{for all } X, Y \in \mathcal{X}(\mathbb{H}_1),$$

where $g_0(X,Y)$ is the inner product on the tangent bundle $T(\mathbb{H}_1)$ induced by the Poincaré ds_0^2. But the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\, ds^2_{1,1;1,1})$ is not an Einstein manifold. In fact, if we denote by $S(X,Y)$ the Ricci curvature of the Siegel-Jacobi space $(\mathbb{H}_1 \times \mathbb{C},\, ds^2_{1,1;1,1})$, we can see without difficulty that there does not exist a constant c such that

$$S(E_1, E_1) = c\, g(E_1, E_1) = c\, g_{11}.$$

4. Final remarks

Let $\mathbb{D} = \{\zeta \in \mathbb{C} \,|\, |\zeta| < 1\}$ be the unit disk in the complex plane. We let

$$G_*^J := \left\{ \left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi, \overline{\xi}; i\,\kappa) \right) \,\Big|\, \begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix} \in SU(1,1),\ \xi \in \mathbb{C},\ \kappa \in \mathbb{R} \right\}$$

be the Jacobi group equipped with the multiplication law

$$\begin{aligned} &\left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi, \overline{\xi}; i\,\kappa) \right) \cdot \left(\begin{pmatrix} p' & q' \\ \overline{q'} & \overline{p'} \end{pmatrix}, (\xi', \overline{\xi'}; i\,\kappa') \right) \\ = &\left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix} \begin{pmatrix} p' & q' \\ \overline{q'} & \overline{p'} \end{pmatrix}, (\tilde{\xi} + \xi', \tilde{\theta} + \overline{\xi'};\, i\,\kappa + i\,\kappa' + \tilde{\xi}\,{}^t\overline{\xi'} - \tilde{\theta}\,{}^t\xi') \right), \end{aligned}$$

where $\tilde{\xi} = p'\xi + \overline{q}\,\overline{\xi}$ and $\tilde{\theta} = q'\xi + \overline{p'}\,\overline{\xi}$. Then G_*^J acts on the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ transitively by

$$\left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, (\xi, \overline{\xi}; i\,\kappa) \right) \cdot (\zeta, \eta) = \left(\frac{p\,\zeta + q}{\overline{q}\,\zeta + \overline{p}}, \frac{\eta + \xi\,\zeta + \overline{\xi}}{\overline{q}\,\zeta + \overline{p}} \right), \tag{4.1}$$

where $\zeta \in \mathbb{D}$ and $\eta \in \mathbb{C}$. According to (1.2), we see that $G_{1,1}^J$ acts on $\mathbb{H}_1 \times \mathbb{C}$ transitively by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu; \kappa) \right) \cdot (z, w) = \left(\frac{az + b}{cz + d}, \frac{w + \lambda\, z + \mu}{cz + d} \right), \tag{4.2}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R}),\ \lambda, \mu, \kappa \in \mathbb{R},\ z \in \mathbb{H}_1$ and $w \in \mathbb{C}$.

In [15], the author proved that the action (4.1) of G_*^J on the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ is compatible with the action (4.2) of G_*^J on the Siegel-Jacobi space

$\mathbb{H}_1 \times \mathbb{C}$ via the partial Cayley tranform $\Phi_* : \mathbb{D} \times \mathbb{C} \longrightarrow \mathbb{H}_1 \times \mathbb{C}$ defined by

$$\Phi_*(\zeta, \eta) := \left(\frac{i\,(1+\zeta)}{1-\zeta}\,, \frac{2\,i\,\eta}{1-\zeta} \right), \qquad (\zeta, \eta) \in \mathbb{D} \times \mathbb{C}. \tag{4.3}$$

Precisely, if $g = \left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), (\lambda, \mu; \kappa) \right) \in G^J_{1,1}$, we put

$$g_* = \left(\begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}, \left(\frac{1}{2}\,(\lambda + i\,\mu), \frac{1}{2}\,(\lambda - i\,\mu)\,; -i\,\frac{\kappa}{2} \right) \right), \tag{4.4}$$

where

$$p = \frac{1}{2}\,\{\,(a+d) + i\,(b-c)\}$$

and

$$q = \frac{1}{2}\,\{\,(a-d) - i\,(b+c)\}\,.$$

We note that g_* is an element of G^J_*. The compatibility condition means that the following condition

$$g\cdot\Phi_*(\zeta,\eta) = \Phi_*\big(g_*\cdot(\zeta,\eta)\big) \qquad \text{for all } g \in G^J_{1,1} \text{ and } (\zeta,\eta) \in \mathbb{D} \times \mathbb{C} \tag{4.5}$$

holds. Using the compatibility condition (4.5), the author [16] proved that for any two positive real numbers A and B,

$$\begin{aligned} d\tilde{s}^2_{1,1;A,B} = &\ 4\,A\,\frac{d\zeta\, d\overline{\zeta}}{\big(1-|\zeta|^2\big)^2} \\ &+ 4\,B \left\{ \frac{d\eta\, d\overline{\eta}}{1-|\zeta|^2} + \frac{\big(1+|\zeta|^2\big)\,|\eta|^2 - \overline{\zeta}\,\eta^2 - \zeta\,\overline{\eta}^2}{\big(1-|\zeta|^2\big)^3}\, d\zeta\, d\overline{\zeta} \right. \\ &\qquad \left. + \frac{\eta\,\overline{\zeta} - \overline{\eta}}{\big(1-|\zeta|^2\big)^2}\, d\zeta\, d\overline{\eta} + \frac{\overline{\eta}\,\zeta - \eta}{\big(1-|\zeta|^2\big)^2}\, d\overline{\zeta}\, d\eta \right\} \end{aligned}$$

is a Riemannian metric on the Siegel-Jacobi disk $\mathbb{D} \times \mathbb{C}$ which is invariant under the action (4.1) of G^J_* on $\mathbb{D} \times \mathbb{C}$. According to Theorem 1.4 in [16], we see that the Laplace-Beltrami operator $\widetilde{\Delta}_{1,1;A,B}$ of the Siegel-Jacobi disk $\big(\mathbb{D} \times \mathbb{C},\, d\tilde{s}^2_{1,1;A,B}\big)$ is given by

$$\begin{aligned} \widetilde{\Delta}_{1,1;A,B} = &\ \frac{1}{A} \left\{ \big(1-|\zeta|^2\big)^2\, \frac{\partial^2}{\partial\zeta\,\partial\overline{\zeta}} + \big(1-|\zeta|^2\big)(\eta - \overline{\eta}\,\zeta)\, \frac{\partial^2}{\partial\zeta\,\partial\overline{\eta}} \right. \\ &\qquad + \big(1-|\zeta|^2\big)(\overline{\eta} - \eta\,\overline{\zeta})\, \frac{\partial^2}{\partial\overline{\zeta}\,\partial\eta} \\ &\qquad \left. + \big(|\eta|^2 + |\zeta\,\eta|^2 - \overline{\zeta}\,\eta^2 - \zeta\,\overline{\eta}^2\big)\, \frac{\partial^2}{\partial\eta\,\partial\overline{\eta}} \right\} \\ &+ \frac{1}{B}\,\big(1-|\zeta|^2\big)\, \frac{\partial^2}{\partial\eta\,\partial\overline{\eta}}\,. \end{aligned}$$

Theorem 4.1. *The scalar curvature of the Siegel-Jacobi disk* $\big(\mathbb{D}\times\mathbb{C},\, d\tilde{s}^2_{1,1;A,B}\big)$ *is*

$$r(q) = -\frac{3}{A} \qquad \text{for all } q \in \mathbb{D}\times\mathbb{C}.$$

Proof. The proof follows from Theorem 3.2 and the compatibility condition (4.5). □

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